Appendix

G

Significant Figures and Rounding Off[1]

G.1 Working with Numbers

In most cases, rather than using paper and pencil, you will be performing math calculations using a handheld calculator, a personal computer, or a mainframe computer. However, whether you use an electronic tool or paper and pencil, you are often required to make decisions based on certain basic rules and principles of mathematics. In addition, when a calculator or computer is used, you have the additional responsibility for ensuring that the tool (hardware or software) is, in fact, providing accurate and reliable results.

In this initial lesson some of the most basic mathematical concepts are reviewed. These concepts, though basic and supposedly simple, often lead to periods of frustration and hair pulling when ignored or overlooked. The basics to be presented in this lesson deal with determining how many figures to keep (where to truncate) and how or when to round off.

The number of digits displayed as the answer on most calculators and computers is governed by the physical properties of the instrument (e.g., many handheld calculators display only 10 digits). Determining how many digits to keep (where to truncate, or which digits to throw away), and when and how to round are decisions that you must make. On the next page you will be asked to solve 12 problems requiring you to determine which digits to keep and when to round.

Using your calculator, provide your answers in two forms, the *complete* answer, and the *corrected* answer. In the first column (Complete) supply the complete answer obtained by performing the required function (probably an answer with 10 digits on many calculators). In the second column (Corrected), supply the answer retaining the *correct* number of digits, rounded where necessary.

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[1] The information in this appendix is extracted from the EPA's APTI Self Instruction Course (SI:100) found at: http://yosemite.epa.gov/oaqps/EOGtrain.nsf/DisplayView/SI_100_2?OpenDocument.

Intro Problems

	Problem	Complete	Corrected
1.	3.5 + 2.075 =		
2.	3.49 - 2.0075 =		
3.	2.0 x 307 =		
4.	2.49 x 3.07 =		
5.	2.074 x 4.700 =		
6.	4.1 x 3.29875 =		
7.	50 ÷ 3.0069=		
8.	9.4 ÷ 334 =		
9.	9.4000 ÷ 0.02=		
10.	0.052 ÷ 0.0026 =		
11.	0.00791 ÷ 0.52 =		
12.	0.0025 x 0.00025 =		

Now, compare your answers with those provided on page G-10.

G.2 Approximate Numbers

Any number may be classified as exact or as approximate. An exact number is derived from the use of specific numbering systems and arithmetic rules. (For example, 12 is an exact number.) Approximate numbers are derived from measurements and calculations where rounding has been, or may be, applied. When it is stated that 12 eggs are consumed by five people, or that each person consumed 2.4 eggs, 2.4 eggs represents an approximate number. Even if the eggs were scrambled, we have no way of ensuring that each person consumed exactly 2.4 eggs.

With air pollution problems, we deal primarily with measurements. Therefore, we are dealing primarily with approximate numbers. Another way of considering approximate numbers is to acknowledge that an approximate number has some degree of error associated with it. Since the numbers being used are approximate and contain some degree of error at the outset, care must be taken to avoid introducing any more error into problems and their solutions. The following general rules are useful in remembering the rules and calculating the values associated with approximate numbers.

Rule 1 (General)
In most cases, mathematical rules governing the results of an addition or subtraction operation are quite similar to one another, if not the same. Also, the rules governing the results of a multiplication or division operation are similar to one another, or the same. But the rules governing the results of addition and subtraction operations are generally quite different from the rules governing the results of multiplication and division operations.

Rule 2 (General)
When performing calculations with approximate numbers, carry as many digits as possible until the final result is calculated. Once the final result is calculated, apply the appropriate rules for truncating and rounding.

Since the rule for rounding approximate numbers applies to addition, subtraction, multiplication, and division, and is easy to remember, we will look at it first.

G.3 Rounding Approximate Numbers

For the moment, we will not concern ourselves with where and how to truncate numbers. We will simply assume that the appropriate number of digits to be retained are givens in the following examples. When truncating (removing final, unwanted digits), rounding is normally applied to the last digit to be kept.

Rule for Rounding Approximate Numbers

If the value of the first digit to be discarded is less than 5, retain the last kept digit with no change. If the value of the first digit to be discarded is 5 or greater, increase the last kept digit's value by one.

Example: 25.0847
Assume only the first two decimal places are to be kept (the 4 and 7 are to be dropped). Round to 25.08. Since the first digit to be discarded (4) is less than 5, the 8 is not rounded up.

Example: 25.0867
Assume only the first two decimal places are to be kept (the 6 and 7 are to be dropped). Round to 25.09. Since the first digit to be discarded (6) is 5 or more, the 8 is rounded up to 9.

G.4 Adding and Subtracting Approximate Numbers

When adding or subtracting approximate numbers, a rule based upon precision determines how many digits are kept. In general, precision relates to the decimal significance of a number. When a measurement is given as 1.005 cm, we can say that the number is precise to the *thousandth* of a centimeter. If the decimal is removed (1005 cm) we have a number that is precise to *thousands* of centimeters.

You may make a measurement in gallons or liters. Although a gallon or a liter may represent an exact quantity, the measuring instruments that are used are capable of producing approximations only. Using a standard graduated flask as an example, can you determine whether there is exactly one liter? Likely not. In fact you would be hard pressed to verify that there was a liter to within ±1/10 of a liter. Therefore, depending upon the instruments used, the precision of a given measurement may vary.

If a measurement is given to us as 16.0 L, the zero after the decimal indicates that the measurement is precise to within 1/10 L. Given a measurement of 16.00 L, we have precision to 1/100 L. In short, the digits following the decimal indicate how precise the measurement is. Precision is used to determine where to truncate when approximate numbers are added or subtracted.

Truncating Approximate Numbers Following Addition or Subtraction

When approximate numbers are added or subtracted the results are expressed in terms of the least precise number in the problem.

Since this is a relatively simple rule to master, just one problem will be used to illustrate it. Calculate the following and express the result in precise terms:

$$6.04\text{L} + 2.8\text{ L} - 4.173\text{ L} = 4.7\text{ L}$$

The complete result is 4.667 L. The answer follows the rule of precision. The expressions in the problem have two, one, and three decimal places respectively. The least precise number (least decimal places) in the problem is 2.8, a value carried only to the tenths position. Therefore, the answer must be calculated to the tenths position only. Thus the correct answer is 4.7 L. (The last 6 and the 7 are dropped from 4.667 L, and the first 6 is rounded up to provide 4.7 L.)
Intro Problems 1 and 2 represent addition and subtraction of approximate numbers.

Problem 1: 3.5 + 2.075 = 5.575 = 5.6

The least precise number (3.5) is provided to one decimal place. The answer must therefore contain only one decimal and the second 5 is rounded up to 6.

Problem 2: 3.49 - 2.0075 = 1.4825 = 1.48

Two decimal places are represented by the least precise number (3.49). The answer is given to two decimals and the 8 is not rounded up.

G.5 Multiplying and Dividing Approximate Numbers

In multiplication and division of approximate numbers, finding the number of significant digits is used to determine how many digits to keep (where to truncate). We must first understand significant digits in order to determine the correct number of digits to keep or remove in multiplication and division problems.

Significant Digits

Generally, the digits 1 through 9 are considered to be significant. Thus, the numbers 123, 53, 7492, and 5 contain three, two, four, and one significant digits respectively.

The digit 0 must be considered separately.

Zeros are significant when they occur between significant digits. In the following examples, all zeros are significant: 10001, 402, 1.1001, 50.09 (five, three, four, and four significant digits respectively).

Zeros are not significant when they are used as place holders. When used as a place holder, a zero simply identifies where a decimal is located. For example. each of the following numbers has only one significant digit: 1000, 500, 60, 0.09, 0.0002. In the numbers 1200, 540, and 0.0032, there are two significant digits, and the zeros are not significant.

When zeros follow a decimal and are preceded by a significant digit, the zeros are significant. In the following examples, all zeros are significant: 1.00, 15.0, 4.1000, 1.90, 10.002, 10.0400. In the example 10.002, the zeros are significant because they fall between two significant digits. In the last example, 10.0400, the first two zeros are significant because they fall between two significant digits; the last two zeros are significant because they follow a decimal and are preceded by a significant digit.

Additional illustrations of significant digits are provided in the following chart. The significant digits are underlined.

Example		Number of Significant Digits
1.	123	3
2.	12300	3
3.	12003	5
4.	123.000	6
5.	12300.0	6
6.	1.0004	5
7.	0.0004	1
8.	0.005003	4
9.	0.005300	4
10.	1000.0001	8

Example 1 is pretty easy. There are three non-zero digits and no decimal places; therefore, three significant digits. Example 2 uses two zeros as "place holders" to locate the decimal. The two zeros are not significant; thus, only three digits are significant. In example 3 the two zeros are not place holders, but part of a five-digit number; hence, five significant digits. Example 4 contains three zeros after the decimal. The zeros follow a decimal and are preceded by three significant digits. (The zeros show precision, which is explained later.) Example 5 is similar to the previous example. By the presence of the zero after the decimal preceded by significant digits, the last zero becomes significant. Now the two zeros before the decimal become significant since they fall between significant digits.

The three zeros in example 6 follow the rule described in examples 4 and 5. The zeros in example 7 establish the position of the decimal only; therefore, they are not significant and the 4 is the lone significant digit. Example 8 uses four zeros. The first two zeros (place holders) are not significant; the other two are significant digits. In example 9, the two trailing zeros are significant because they follow a significant digit that follows a decimal. In the last example, all six zeros are significant since they all fall between significant digits.

Having determined how to count significant digits, we can now apply this information to determine where to truncate the results from multiplying or dividing approximate numbers.

Truncating Approximate Numbers Following Multiplication or Division

When approximate numbers are multiplied or divided, the result is expressed as a number having the same number of significant digits as the expression in the problem having the least number of significant digits.

In other words, if you multiply a number having four significant digits by a number having two significant digits, the correct answer will be expressed to two significant digits.

Let's consider a measurement of 200 ft. Not knowing how the measurement was made, we can only know for certain that the measurement represents a distance of 200 ft or greater but less than 300 ft. There is one significant digit, and no matter what computation this measurement enters, the result is good to only one significant digit. Thus, if the problem 200 ft x 13.6 is solved, the complete answer is 2720.0 ft. The two numbers, 200 and 13.6, represent one and three significant digits, respectively. One significant digit is less than three; therefore the correct answer will be rendered to one significant digit. Thus, after rounding, the correct answer is 3000 ft.

If the measurement were made to two significant digits, such as 290 ft, we know that the measurement represents a distance of 290 ft or greater, but less than 300 ft. Again using the measurement, 290 ft x 13.6, the complete result yields 3944.0, and the correct result is 3900 ft. In this case, two significant digits are used (39). Since the first discarded digit is 4, the 39 remains.

Now let's reconsider the answers to problems 3 through 12 (on page 9) for the problems you worked.

Problem 3: 2.0 x 307 = 614 = 610
The number 2.0 represents two significant digits since the zero following the decimal follows a significant digit. The number 307 has three significant digits. The *least* number of significant digits is two. Therefore, the 4 in the answer is not significant and it is less than 5, so the answer, properly rounded to two significant digits, is 610. The 4 is dropped.

Problem 4: 2.49 x 3.07 = 7.6443 = 7.64
There are three significant digits in each number of the problem. The answer, expressed to three significant digits, is 7.64, keeping the decimal and dropping the two nonsignificant digits (43).

Problem 5: 2.074 x 4.700 = 9.7478 = 9.748
Again, both numbers in the problem have the same number of significant digits (four). By keeping four significant digits (9.747), truncating the 800 and rounding, we have 9.748.

Problem 6: 4.1 x 3.29875 = 13.524875 = 14
The numbers in this problem represent two and six significant digits, respectively. Using the fewest significant digits (two) the 13 is kept. By dropping and rounding 0.524875, the correct result is 14.

Problem 7: 50 ÷ 3.0069 = 16.6284213 = 20
The first number, 50, has one significant digit. Therefore, the results will be expressed with the accuracy of one significant digit. The first digit to be truncated is the first 6. So, rounding the 1, the only significant digit becomes 2. The correct answer is then 20.

Problem 8: 9.4 ÷ 3.34 = 2.814371257 = 2.8
Two significant digits divided by three significant digits means the answer must be calculated to two significant digits. The first digit to be discarded is the 1. The 8 remains unchanged, and the answer is 2.8.

Problem 9: 9.4000 ÷ 0.02 = 470 = 500
The 0.02 in the problem contains the least number of significant digits, one. Remember that zeros used as place holders are not significant digits. Therefore, the 4 in the answer must be retained as the only significant digit. By dropping the 7 and rounding, the 4 becomes 5 and the answer is 500.

Problem 10: 0.052 ÷ 0.0026 = 20 = 20
Here, both numbers in the problem are comprised of two significant digits. As it happens, the calculated number and the correct number are the same.

Problem 11: 0.00791 ÷ 0.52 = 0.015211538 = 0.015
Again, the least number of significant digits is two (0.52). Since the first digit to be discarded is a 2, the 5 remains unchanged and the answer is 0.015.

Problem 12: 0.0025 x 0.00025 = 0.000000625 = 0.00000063
Once again, the least number of significant digits is two. The answer yielded three significant digits, 625. The 5 is discarded, the 2 is rounded up to 3, and the answer is 0.00000063.

G.6 Reasonability

The rules for handling approximate numbers are used when there is no overriding rule or condition to be met. In all measurement work, deriving correct answers must be considered in context of the conditions that exist. For example, suppose you are provided numeric data to be processed. The results of your calculations are to be given to a technician who will adjust the airflow through a system. The calculation and raw results are as follows. What value should you give to the technician?

$$20.067 \text{ cfm} \times 12.9362 \text{ cfm} + 18.00782 \text{ cfm} = 14.41544426 \text{ cfm}$$

You probably arrived at the value 14.415 cfm, which is a correct value using the rules provided. However, if the gauge that the technician uses to adjust the airflow is calibrated in whole cubic feet per minute, what value should you provide? The numbers following the decimal are of no value so you should give the technician the value of 14 cfm. So now things are all set, or are they? With the piece of equipment being used, the manufacturer states that it is better to be on the high side rather than on the low side. So, since the mathematical results are actually more than 14 cfm, you had better give the technician a value of 15 cfm.

At this point you should see that even though 14.415 cfm is the correct mathematical result, the solution must be modified by reasonability to meet the environment and the operating conditions.

Rule to Meet the Conditions of Reasonability

This rule of common sense may be stated as follows: *Keep only those results that are reasonable (meaningful) in the context of the work being done and the equipment being used.*

Remember, generally you are not dealing with exact numbers, you are working with approximate numbers. With approximate numbers you must always be careful not to introduce undesirable errors into the final results. When multiple calculations are performed on approximate numbers, errors may become so large that the final results are of no value.

G.7 Practice Exercises

Answers are located on page G-11.

1. Give the number of significant digits for each of the following:

    a. 3.7 = ____________
    b. 2.06 = ____________
    c. 17.41 = ____________
    d. 0.114 = ____________
    e. 0.00134 = ____________
    f. 12000.0 = ____________
    g. 12000 = ____________
    h. 1200.001 = ____________

2. Give the most accurate/precise number for the following calculations:

    a. 1.50 + 2.317 = ____________
    b. 1.50 - 2.317 = ____________
    c. 1500 x 3.94 = ____________
    d. 1500 + 3.94 = ____________
    e. 1.500 + 394 = ____________

G.8 Answers to Intro Problems

	Problem	Complete	Corrected
1.	3.5 + 2.075 =	5575	5.6
2.	3.49 - 2.0075 =	1.4825	1.48
3.	2.0 x 307=	614.0	610
4.	2.49 x 3.07 =	7.6443	7.64
5.	2.074 x 4.700 =	9.7478	9.748
6.	4.1 x 3.29875 =	13.524875	14
7.	50 ÷ 3.0069 =	16.6284213	20
8.	9.4 ÷ 334 =	2.814371257	2.8
9.	9.4000 ÷ 0.02 =	470	500
10.	0.052 ÷ 0.0026 =	20	20
11.	0.00791÷ 0.52 =	0.015211538	0.015
12.	0.0025 x 0.00025 =	0.000000625	0.00000063

The numbers given in the preceding chart represent the answers obtained when the correct rules for truncating and rounding are applied.

G.9 Answers to Practice Exercises

1. Give the number of significant digits for each of the following:

   a. 3.7 = **2**
   b. 2.06 = **3**
   c. 17.41 = **4**
   d. 0.114 = **3**
   e. 0.00134 = **3**
   f. 12000.0 = **6**
   g. 12000 = **2**
   h. 1200.001 = **7**

2. Give the most accurate/precise number for the following calculations:

   a. 1.50 + 2.317 = **3.82**
   b. 1.50 - 2.317 = **-0.82**
   c. 1500 x 3.94 = **5900**
   d. 1500 ÷ 3.94 = **380**
   e. 1.500 ÷ 3.94 = **0.381**